

BOREL CANONIZATION OF ANALYTIC SETS WITH BOREL SECTIONS

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ABSTRACT. Given an analytic equivalence relation, we tend to wonder whether it is Borel. When it is non Borel, there is always the hope it will be Borel on a “large” set – nonmeager or of positive measure. That has led Kanovei, Sabok and Zapletal to ask whether every proper σ ideal satisfies the following property: given E an analytic equivalence relation with Borel classes, there exists a set B which is Borel and I -positive such that $E \upharpoonright_B$ is Borel. We propose a related problem – does every proper σ ideal satisfy: given A an analytic subset of the plane with Borel sections, there exists a set B which is Borel and I -positive such that $A \cap (B \times \omega^\omega)$ is Borel. We answer positively when a measurable cardinal exists, and negatively in L , where no proper σ ideal has that property. Assuming ω_1 is inaccessible to the reals but not Mahlo in L , we construct a ccc σ ideal I not having this property – in fact, forcing with I adds a non Borel section to a certain analytic set with Borel sections, and a non Borel class to a certain analytic equivalence relation with Borel classes. Various counterexamples are given for the case of a Δ_2^1 equivalence relation as well as for the case of an improper ideal.

1. INTRODUCTION

1.1. Borel Canonization of Analytic Equivalence Relations. Analytic equivalence relations are common in the world of mathematics, and given such an equivalence relation, one of the first questions traditionally asked is – “is it Borel?”. A negative answer used to convince us that the equivalence relation is relatively complicated, but a new point of view proposed by Kanovei, Sabok and Zapletal has opened the way to a somewhat more optimistic conclusion. We all know that Lebesgue measurable functions are “almost continuous”, analytic sets are Borel modulo meager sets and colorings of natural numbers are “almost” trivial. We can then hope that even the non Borel analytic equivalence relations are Borel on a substantial set – which leads to the following question:

Problem 1.1. Given an analytic equivalence relation E on a Polish space X , does there exist a positive measure (or non-meager, or uncountable) Borel set B such that E restricted to B is Borel ?

We can use the notion of a σ -ideal to state a more general problem. Given a σ -ideal I , we will say that A is an I -positive set if $A \notin I$, an I -small set if $A \in I$, and a co- I set if $X - A \in I$. The above mentioned problem involved the existence of an I -positive set for the null ideal, the meager ideal and the countable ideal. We restate it for all σ -ideals:

Problem 1.2. Given an analytic equivalence relation E on a Polish space X and a σ -ideal I , does there exist an I -positive Borel set B such that E restricted to B is Borel ?

Unfortunately, that problem has a negative answer, and further assumptions had to be made – both on the equivalence relation E and on the σ -ideal I (see section 4). We recall that for a σ -ideal I , \mathbb{P}_I is the partial order of Borel I -positive subsets, ordered by inclusion. We say that I is *proper* if the associated forcing notion \mathbb{P}_I is proper. Then Kanovei, Sabok and Zapletal have asked the following:

Problem 1.3. [14] *Borel canonization of analytic equivalence relations with Borel classes:* Given an analytic equivalence relation E on a Polish space X , all of its classes Borel, and a proper σ -ideal I , does there exist an I -positive Borel set B such that E restricted to B is Borel?

They have shown the answer to be positive for two important classes of analytic equivalence relations with Borel classes: orbit equivalence relation, and countable equivalence relations (proofs are given in the next section). The problem in its full generality remained open.

1.2. Borel Canonization of Analytic Sets with Borel Sections. Let E be an analytic equivalence relation on X Polish, all of its classes Borel. The equivalence relation E is a subset of X^2 with Borel sections. Given $B \subseteq X$ Borel, $E \upharpoonright B$ is Borel is equivalent, by the very definition, to $E \cap (B \times B)$ being Borel. That simple observation leads to the following variants of Borel canonization:

Definition 1.4. Let X be Polish, and I a σ -ideal on X .

- (1) We say that I has *square Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists an I -positive Borel set B such that $A \cap (B \times B)$ is Borel.
- (2) We say that I has *rectangular Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists an I -positive Borel set B such that $A \cap (B \times X)$ is Borel.
- (3) We say that I has *strong square Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists a co- I Borel set B such that $A \cap (B \times B)$ is Borel.
- (4) We say that I has *strong rectangular Borel canonization of analytic sets with Borel sections* if for any $A \subseteq X^2$ an analytic set with vertical Borel sections, there exists a co- I Borel set B such that $A \cap (B \times X)$ is Borel.

In what follows, we will simply say: “ I has square Borel canonization”, etc. Rectangular Borel canonization implies square Borel canonization, which implies Borel canonization of analytic equivalence relations with Borel classes. We do not know whether any of the inverse implications are true.

Remark 1.5. For ccc ideals, the strong Borel canonization and the weak Borel canonization are equivalent. The strong Borel canonization of general proper ideals is false – see [11] proposition 17.

When considering the square and rectangular Borel canonizations, there is no difference between analytic and coanalytic sets:

Claim 1.6. I has square Borel canonization of analytic sets with Borel sections if and only if I has square Borel canonization of coanalytic sets with Borel sections (and the same for rectangular, strong square and strong rectangular Borel canonizations).

Proof. Consider the complement. □

Hence in that context, analytic sets and coanalytic sets are basically the same object.

Albeit being a new notion, strong rectangular Borel canonization has been studied in the past by Fujita in [6] and by Ikegami in [10] and [11], culminating in the following result:

Theorem 1.7. (Ikegami [11]) *Let I be a Borel generated σ -ideal such that \mathbb{P}_I is strongly arboreal, provably ccc and Σ_1^1 . Then the following are equivalent:*

- (1) I has strong rectangular Borel canonization.
- (2) Σ_2^1 sets are measurable with respect to I , which is: For $A \in \Sigma_2^1$ there is B Borel such that $A \triangle B \in I$.

We say that I is *Borel generated* if any $A \in I$ is contained in an I -small Borel set. We say that I is *provably ccc* if ZFC proves that I is ccc. The notions of “strongly arboreal” and “ Σ_1^1 forcing” will be defined in the following section. For now, we will only say these are assumptions on the presentability and definability of \mathbb{P}_I , satisfied by, for example, the meager ideal and the null ideal. Hence, one learns from the theorem that the meager ideal has strong rectangular Borel canonization if and only if Σ_2^1 sets have the Baire property, and the null ideal has strong rectangular Borel canonization if and only if Σ_2^1 sets are Lebesgue measurable.

This paper will focus on general σ -ideals with minimal assumptions on definability and presentability. It is therefore interesting and illuminating to compare our results with Ikegami’s results.

1.3. The results of this paper. The problem of Kanovei, Sabok and Zapletal can be restated as:

Problem 1.8. Do all proper ideals I have Borel canonization of analytic equivalence relation with Borel classes?

We focus our paper at the following related problem:

Problem 1.9. Do all proper ideals I have rectangular Borel canonization of analytic sets with Borel sections?

Section 2 reviews definitions and facts which we use in this paper, and elaborates on previous results about Borel canonization.

In section 3 we define a notion of ω_1 -rank for analytic sets with Borel sections. We use the rank to prove:

Theorem 1.10. *Assume a measurable cardinal exists. Then proper ideals have rectangular Borel canonization and ccc ideals have strong rectangular Borel canonization.*

We say that ω_1 is *inaccessible to the reals* if for every z real, $\omega_1^{L[z]} < \omega_1$.

Theorem 1.11. *Assume ω_1 is inaccessible to the reals, and I is ccc in $L[z]$ for any real z . Then I has strong rectangular Borel canonization of analytic sets all of whose sections are Π_γ^0 for some $\gamma < \omega_1$.*

Section 4 presents examples and counterexamples, mainly demonstrating the necessity of assuming the properness of I in problem 1.3. For example:

Proposition 1.12. *Let E be analytic with uncountably many classes but not perfectly many. Let I_E be the σ -ideal generated by the equivalence classes. Then for any B Borel I_E -positive, $E \restriction_B$ is non Borel.*

In [5], we show that I_E as above is never proper, hence that proposition does not provide a negative answer to the problem of Borel canonization (problem 1.3).

Extending our discussion to Δ_2^1 sets, we remark that L demonstrates a strong form of failure of Borel canonization:

Theorem 1.13. [3] *In L , σ -ideals do not have Borel canonization of Δ_2^1 equivalence relations with Borel classes.*

The last section presents counterexamples to rectangular Borel canonization, both in L and in much larger universes:

Proposition 1.14. *In L , proper ideals do not have rectangular Borel canonization of analytic sets with Borel sections. The same is true for $L[z]$ where z is a real.*

Theorem 1.15. *If ω_1 is inaccessible to the reals and is not Mahlo in L , then there is a ccc ideal not having rectangular Borel canonization of analytic sets with Borel sections. Moreover, $\mathbb{P}_I \Vdash A_{x_G}$ non Borel for some A analytic with Borel sections.*

Corollary 1.16. *Rectangular Borel canonization for ccc ideals implies that ω_1 is inaccessible to the reals and Mahlo in L .*

Non absoluteness of “all sections / classes are Borel” is further demonstrated by the following proposition:

Proposition 1.17. *There is an analytic equivalence relation E such that:*

- (1) *If ω_1 is inaccessible to the reals and is not Mahlo in L , then all E classes are Borel and there is a ccc ideal I such that*

$$\mathbb{P}_I \Vdash [x_G] \text{ is non Borel.}$$

- (2) *If ω_1 is inaccessible to the reals, then all E classes are Borel, while in L there is a non Borel class.*

The problem of square Borel canonization is sometimes discussed in this paper but the consistency of a negative answer remains open. The same applies for the problem of Borel canonization of equivalence relations.

Chan, in [3], has independently obtained much of the above results using similar techniques. He has been working with equivalence relations, but his proofs perfectly fit in the context of rectangular Borel canonization. In particular, he has shown that all proper ideals have rectangular Borel canonization if there exist sharps for all reals and for a few more sets associated with the forcing notions of proper ideals.

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2. PRELIMINARIES

2.1. Forcing with Ideals. The basics of forcing can be found in [12]. We remind that a forcing notion satisfies the *countable chain condition*, or is *ccc*, when every antichain is countable. A wider class of forcing notions is the proper ones:

Definition 2.1. Let M be an elementary submodel of H_θ , and \mathbb{P} a forcing notion. A condition $q \in \mathbb{P}$ is a *master condition* over M if for every $D \subseteq \mathbb{P}$ dense such that $D \in M$, $q \Vdash \dot{G} \cap \check{D} \cap \check{M} \neq \emptyset$. A forcing notion \mathbb{P} is *proper* if for every θ large enough, every countable elementary submodel M of H_θ such that $\mathbb{P} \in M$, and every $p \in \mathbb{P} \cap M$, there is a $q \leq p$ which is a master condition over M .

Proper forcing notions preserve ω_1 , but there are ω_1 -preserving forcing notions which are not proper.

The subject of forcing with ideals was thoroughly investigated by Zapletal in [19]. We review here some of the most important notions and facts.

Recall that for I a σ -ideal on a Polish space X , \mathbb{P}_I is the partial order of Borel I -positive sets ordered by inclusion.

Proposition 2.2. *The poset \mathbb{P}_I adds an element x_G of the Polish space X such that for every Borel set $B \subseteq X$ coded in the ground model, $B \in G \Leftrightarrow x_G \in B$.*

Given M a transitive model of ZFC , we say that $x \in X$ is generic over M if

$$\{B \in \mathbb{P}_I \cap M : x \in B\}$$

is a generic filter over M .

A \mathbb{P}_I -generic point avoids all Borel I -small sets of the ground model. When \mathbb{P}_I is ccc, this is a complete characterization of the generic points:

Proposition 2.3. *Let M be a transitive model of ZFC , and $I \in M$ a σ -ideal on X such that $M \models \mathbb{P}_I$ is ccc. Then $x \in X$ is \mathbb{P}_I generic over M if and only if for every I -small set B coded in M , $x \notin B$.*

Common in this paper is forcing over a model which is well founded but not transitive – what we really mean by that is forcing over its transitive collapse.

Corollary 2.4. *Let I be a σ -ideal such that \mathbb{P}_I is ccc, and M a countable elementary submodel of a large enough H_θ such that $\mathbb{P}_I \in M$ and $B \in \mathbb{P}_I \cap M$. The set of elements of B which are generic over M is co- I in B .*

We say that I is ccc if \mathbb{P}_I is ccc, and that I is proper if \mathbb{P}_I is proper. Properness of \mathbb{P}_I can be phrased in terms of the set of M -generics:

Proposition 2.5. *\mathbb{P}_I is proper if and only if for every M a countable elementary submodel of a large enough H_θ such that $\mathbb{P}_I \in M$ and for every $B \in \mathbb{P}_I \cap M$, the set of elements of B which are generic over M is I -positive.*

As an example, we use the above characterization to show:

Corollary 2.6. *Forcing with a proper ideal preserves ω_1 .*

Proof. Assume otherwise, and fix $B \in \mathbb{P}_I$ such that

$$B \Vdash \dot{f} : \omega \rightarrow \omega_1 \text{ onto.}$$

Let M be a countable elementary submodel of a large enough H_θ such that $\mathbb{P}_I \in M$ and $B \in M$. Let $C \subseteq B$ be the I -positive Borel set of the M -generics of B . Now force an $x_G \in C$ which is \mathbb{V} -generic. In $\mathbb{V}[x_G]$, x_G is still an element of C , and C is defined as the set of M -generics, so x_G is M -generic. At the same time, $\mathbb{V}[x_G]$ interprets \dot{f} as a map from ω onto $\omega_1^\mathbb{V}$. The interpretation of \dot{f} in $M[x_G]$ should be consistent with the one of $\mathbb{V}[x_G]$. Since both models have the same interpretation of ω , $\dot{f}^{M[x_G]} = \dot{f}^{\mathbb{V}[x_G]}$, and in particular $M[x_G]$ contains $\omega_1^\mathbb{V}$. But $M[x_G]$ and M have the same ordinals, and M is countable – a contradiction. \square

This paper is about Borel canonization, and we make an informal claim that Borel canonization and genericity over M are strongly connected. Intuitively, the generic elements over M are well described by the countably many conditions in $\mathbb{P}_I \cap M$, so one can hope that restricting equivalence relations to the set of M -generics will make the equivalence relation more definable. The above propositions assure that when I is proper, the set of generics is indeed big: I -positive in general and co- I for ccc ideals. As a result, when I is proper, for Borel canonization it will be enough to show that the equivalence relation is simpler on the set of generics. For improper ideals, a completely different approach should probably be taken.

2.2. Borel canonization of orbit equivalence relations and countable equivalence relations. We now give the proofs of the two Borel canonization results of Kanovei, Sabok and Zapletal [14]. The first one is rewritten using the notion of Hjorth rank (see [9, 4]), and the second one is generalized so that it shows rectangular Borel canonization of analytic sets with countable sections:

Theorem 2.7. *Proper ideals have Borel canonization of orbit equivalence relations.*

Proof. Let G be a Polish group acting on a Polish space X , and I a proper σ -ideal. We find C Borel and I -positive such that $(E_G^X) \upharpoonright_C$ is Borel. Let δ be the Hjorth rank associated with the action of G on X . Fix θ large enough and $M \preceq H_\theta$ an elementary submodel containing all the relevant information. Let C be the I -positive Borel set of M -generics, and let $x \in C$ be M -generic. Then

$$M[x] \models \delta(x) \leq \alpha$$

for some $\alpha < \omega_1^{M[x]} = \omega_1^M$. The rank δ has a Borel definition, hence $\mathbb{V} \models \delta(x) \leq \alpha$ as well. We have thus proved that the Hjorth rank on C is uniformly bounded below ω_1^M , hence $(E_G^X) \upharpoonright_C$ is Borel. \square

Theorem 2.8. *Proper ideals have rectangular Borel canonization of analytic sets with countable sections.*

Proof. Fix I proper and A an analytic subset of the plane with countable sections. Recall that a $\Sigma_1^1(x)$ set is countable if and only if all its elements are hyperarithmetic in x . One can then show that “all sections are countable” is still true in generic extensions. Use 2.3.1 of [19] to find $B \in \mathbb{P}_I$ and a Borel $f : B \rightarrow X^\omega$ such that $B \Vdash f(x_G)$ enumerates A_{x_G} .

Fix θ large enough and $M \preceq H_\theta$ an elementary submodel containing all the relevant information (including f and B). Let $C \subseteq B$ be the I -positive Borel set of M -generics, and let $x \in C$ be M -generic. Then

$$M[x] \models f(x) \text{ enumerates } A_x,$$

which is,

$$M[x] \models \forall y (y \in A_x \Rightarrow \exists n \in \omega (f(x))(n) = y).$$

That statement is Π_1^1 , so it must be true in \mathbb{V} as well – which is, $(A_x)^\mathbb{V} \subseteq M[x]$. On the other hand, if

$$(f(x))(n) = y$$

then $y \in M[x]$ and $M[x] \models y \in A_x$, hence \mathbb{V} thinks the same.

The above results in a Borel definition of $A \cap (C \times X)$: For $x \in C$ and $y \in X$,

$$(x, y) \in A \Leftrightarrow \exists n \in \omega (f(x))(n) = y. \quad \square$$

2.3. \mathbb{P}_I -measurable sets. We quickly review the definitions and results of [10].

A *tree on ω* is a subset of $\omega^{<\omega}$ closed under initial segments. The set of branches through T is

$$[T] = \{f \in \omega^\omega : \forall n f \upharpoonright_n \in T\}.$$

For a forcing notion \mathbb{P} , we say that \mathbb{P} is *strongly arboreal* if the conditions of \mathbb{P} are perfect trees on ω , and

$$T \in \mathbb{P}, s \in T \Rightarrow T \upharpoonright_s \in \mathbb{P}$$

where $T \upharpoonright_s = \{t : t \in T; t \supseteq s \text{ or } t \subseteq s\}$. A strongly arboreal forcing notion adds a generic real x_G such that $\mathbb{V}[x_G] = \mathbb{V}[G]$. The generic x_G is exactly that real in $\mathbb{V}[G]$ that is a branch through all trees of the generic filter G . We will abuse notation and say that \mathbb{P} is strongly arboreal if it has a presentation that is strongly arboreal.

Let \mathbb{P} be strongly arboreal. A set of reals A is \mathbb{P} -null if every $T \in \mathbb{P}$ can be extended to T' such that $[T'] \cap A = \emptyset$. We denote by $N_{\mathbb{P}}$ the collection of \mathbb{P} -null sets, and by $I_{\mathbb{P}}$ the σ -ideal generated by the \mathbb{P} -null sets.

We say that a set of reals A is \mathbb{P} -measurable if every $T \in \mathbb{P}$ can be extended to T' such that $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] - A \in I_{\mathbb{P}}$.

Given a ccc ideal I such that \mathbb{P}_I is strongly arboreal, one can consider all the above notions for \mathbb{P}_I . The resulting σ -ideal $I_{\mathbb{P}_I}$ will be the one generated by the Borel I -small sets. Hence when I is Borel generated, $I = I_{\mathbb{P}_I}$. In that case, \mathbb{P}_I -measurable is what is usually called I -measurable (see [2]).

A σ -ideal I is said to be Σ_n^1 or Π_n^1 if the set of Borel codes of I -small sets is. The term “provably ccc” refers to σ -ideals which are ccc in all models of ZFC .

Theorem 2.9. *Let I be a provably ccc, provably Δ_2^1 and Borel generated σ -ideal such that \mathbb{P}_I is strongly arboreal. The following are equivalent:*

- (1) *Every Δ_2^1 set of reals is \mathbb{P}_I -measurable.*
- (2) *For any real a and $T \in \mathbb{P}$, there is an $L[a]$ generic in $[T]$.*

Theorem 2.10. *Let I be a provably ccc, provably Δ_2^1 and Borel generated σ -ideal such that \mathbb{P}_I is strongly arboreal. The following are equivalent:*

- (1) *Every Σ_2^1 set of reals is \mathbb{P}_I -measurable.*
- (2) *For any real z , the set of \mathbb{P}_I generics over $L[z]$ is co- I .*

The above were used in [11] to obtain:

Theorem 2.11. *Let I be a provably ccc, Σ_1^1 and Borel generated σ -ideal such that \mathbb{P}_I is strongly arboreal. Then the following are equivalent:*

- (1) *I has strong rectangular Borel canonization.*
- (2) *Every Σ_2^1 set of reals is \mathbb{P}_I -measurable.*

In fact, Ikegami has shown the following:

Lemma 2.12. *Let A be a $\Sigma_1^1(z)$ subset of the plane with Borel sections. Then for B a Borel subset of $L[z]$ -generics, $A \cap (B \times \omega^\omega)$ is Borel.*

Lemma 2.12 together with theorem 2.10 proves (2) \Rightarrow (1) of theorem 2.11.

3. RANKS FOR ANALYTIC SETS WITH BOREL SECTIONS

Let A be an analytic subset of $(\omega^\omega)^2$. There exists a tree $T \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ such that

$$(x, y) \in A \Leftrightarrow T_{xy} \notin WF.$$

For $\alpha < \omega_1$, define:

$$(x, y) \in A_\alpha \Leftrightarrow T_{xy} \notin WF_\alpha.$$

The sequence A_α is decreasing, $A_\delta = \bigcap_{\alpha < \delta} A_\alpha$ for δ limit, and

$$A = \bigcap_{\alpha < \omega_1} A_\alpha.$$

Definition 3.1. For $x \in \omega^\omega$, the *rank* of x , $\delta(x)$, is the least α such that $A_x = (A_\alpha)_x$, if such an α exists, and ∞ if there is no such α .

Proposition 3.2. *If A_x is Borel, then there is $\alpha < \omega_1$ such that $A_x = (A_\alpha)_x$.*

Proof. Since

$$(\sim A)_x = \{y : (x, y) \notin A\} = \{y : T_{xy} \in WF\}$$

is a Borel set, its image under $y \rightarrow T_{xy}$ is an analytic subset of WF . By the boundedness theorem for WF , its image is contained in WF_α for some countable α , which is:

$$y \in A_x \Leftrightarrow T_{xy} \notin WF_\alpha \Leftrightarrow y \in (A_\alpha)_x$$

as we wanted to show. \square

Proposition 3.3. *The set $\Delta = \{(x, f) : f \in WO, \delta(x) \leq ot(f)\}$ is Π_2^1 . The set $\{x : A_x \text{ Borel}\}$ is Σ_3^1 .*

Proof. $f \in WO$ is Π_1^1 . The rank of x is less than the order type of f if and only if

$$\forall z : T_{xz} \in WF \Leftrightarrow T_{xz} \in WF_{ot(f)}$$

which is Π_2^1 . For $x \in X$, A_x is Borel if and only if $\exists f$ such that $(x, f) \in \Delta$, which is Σ_3^1 . \square

Proposition 3.4. *Let $B \subseteq X$ be a Borel set. Then $A \cap (B \times X)$ is Borel if and only if there is an $\alpha < \omega_1$ such that for all $x \in B$, $\delta(x) < \alpha$.*

The proof uses the boundedness theorem for WF in the same way used in the proof of proposition 3.2.

When one considers square Borel canonization, or Borel canonization of equivalence relations, the rank δ has to be relativized:

Definition 3.5. For $x \in \omega^\omega$ and B Borel, the rank of x with respect to B , $\delta_B(x)$, is the least α such that $A_x \cap B = (A_\alpha)_x \cap B$, if such an α exists, and ∞ if there is no such α .

Proposition 3.6. *Let $B \subseteq X$ be a Borel set. The set $\Delta = \{(x, f) : f \in WO, \delta_B(x) \leq ot(f)\}$ is Π_2^1 . The set $\{x : A_x \cap B \text{ is Borel}\}$ is Σ_3^1 .*

Proposition 3.7. *Let $B \subseteq X$ be a Borel set. Then $A \cap (B \times B)$ is Borel if and only if there is an $\alpha < \omega_1$ such that for all $x \in B$, $\delta_B(x) < \alpha$.*

We remark that the rank is not canonical and depends on the choice of the tree T . However, all we will need for our results is the mere existence of such a rank.

Example 3.8. Analytic equivalence relations with Borel classes are convenient examples of analytic subsets of the plane with Borel sections. Let us find a rank for two such equivalence relations:

- (1) Given $x, y \in LO$ linear orders:

$$xE_{\omega_1}y \Leftrightarrow (x, y \notin WO) \vee (ot(x) = ot(y)).$$

We fix a tree T inducing E_{ω_1} : (x, y, f) is in T if and only if either f is an isomorphism between x and y or f codes two ω -decreasing sequences – one in x and the other in y . It is not hard to show that $x \in WO_\alpha \Rightarrow \delta(x) = \alpha$, and $x \notin WO \Rightarrow \delta(x) = \infty$.

- (2) Let $A \subseteq \omega^\omega$ be a strictly analytic set. Given $(x_1, x_2), (y_1, y_2) \in \omega^\omega \times \{0, 1\}$, $(x_1, x_2)E(y_1, y_2)$ if and only if

$$(x_1 = y_1 \in A) \vee ((x_1, x_2) = (y_1, y_2))$$

The equivalence relation E is strictly analytic and all its classes are finite. Fix a tree T such that

$$x \in {}^\sim A \Leftrightarrow T_x \in WF$$

and for each α countable, let

$$x \in B_\alpha \Leftrightarrow T_x \in WF_\alpha.$$

The sets B_α are Borel and ${}^\sim A = \bigcup_{\alpha < \omega_1} B_\alpha$.

We can now use the tree T to define another tree inducing the equivalence relation E , and consider the rank associated with the new tree. It is then easy to see that $x \in B_\alpha \Rightarrow \delta(x) = \alpha$, and $x \in A \Rightarrow \delta(x) = \omega$.

That example shows that the rank might be arbitrarily higher than the complexity of the equivalence class.

3.1. Rectangular Borel canonization of Proper ideals. Having those definitions in mind, one can try and prove rectangular Borel canonization of proper ideals in the following way:

- Fix a countable elementary submodel $M \preceq H_\theta$ for θ large enough, and force with \mathbb{P}_I over M .
- Show that A_{x_G} is Borel in $M[x_G]$ and so

$$M[x_G] \models \delta(x_G) \leq \alpha$$

for some $\alpha < \omega_1^{M[x]} = \omega_1^M$ (recall that \mathbb{P}_I preserves ω_1).

- Use absoluteness to show that $\mathbb{V} \models \delta(x_G) \leq \alpha$.
- Use properness to guarantee that the set of M -generics is I -positive, and the above arguments to conclude that all of them has rank less than $\omega_1^M < \omega_1$.

However, the 2nd and 3rd steps are in general impossible. Although A has only Borel sections, that statement is Π_4^1 (see proposition 3.3), hence one must work harder to show its preservation. The 3rd step provides us with another absoluteness challenge, since Π_2^1 absoluteness between a submodel N and the universe is guaranteed when N contains all countable ordinals, whereas $M[x_G]$ is countable.

The following proof follows the above lines and takes advantage of the measurable cardinal to overcome the above mentioned difficulties. We remind that by a theorem of Martin and Solovay (15.6 in [13]), when there is a measurable cardinal κ , forcing notions of cardinality less than κ preserve Σ_3^1 statements.

Theorem 3.9. *Assume a measurable cardinal exists. Then proper ideals have rectangular Borel canonization and ccc ideals have strong rectangular Borel canonization.*

Proof. The idea is as follows: given U a κ -complete ultrafilter on κ , one can form iterated ultrapowers of the universe, \mathbb{V}_α , all well founded by a theorem of Gaifman. The same operation can be applied on M a countable elementary submodel of the universe such that $U \in M$. Since the sequence $j^{(\alpha)}(\kappa)$ is increasing and continuous, M_{ω_1} , the ω_1 'th iterated ultrapower of M , contains all countable ordinals, so that M_{ω_1} and the universe agree on Π_2^1 statements. On the other hand, M_{ω_1} is an iterated ultrapower of M , so they agree on all statements – there is an elementary embedding between them. We will then have enough absoluteness to conclude the proof.

So let $M \preceq H_\theta$ for θ large enough be a countable elementary submodel such that $\kappa \in M$ is measurable and M contains all the relevant information. Fix $U \in M$ a κ -complete ultrafilter on κ , and force with \mathbb{P}_I over M . Levy-Solovay theorem guarantees that U remains a κ -complete ultrafilter in $M[x_G]$. For convenience, denote $M[x_G]$ by N , remembering that $\omega_1^N = \omega_1^M$ because \mathbb{P}_I is proper. We can then use U to iterate ultrapowers of

both \mathbb{V} and N over all ordinals. Denote by \mathbb{V}_α and N_α the α 'th iterated ultrapowers of \mathbb{V} and N , respectively. The V'_α 's are well-founded, and since $N_\alpha \subseteq \mathbb{V}_\alpha$, the N'_α 's are well founded as well, so we identify them with their transitive collapses. Since $(j_\alpha)^N(\kappa)$ is a normal sequence, N_{ω_1} has all countable ordinals. Hence, as stated above, N_{ω_1} and N are elementarily equivalent, and N_{ω_1} and \mathbb{V} are Π_2^1 equivalent.

By the assumption, $\mathbb{V} \models A_{x_G} \text{ Borel}$, and so there is a countable ordinal α such that $\mathbb{V} \models \delta(x_G) \leq \alpha$. We would like this statement to be true in N_{ω_1} , but it is meaningless there: Although α is an element of N_{ω_1} , it is not necessarily countable in N_{ω_1} . The natural solution will be collapsing α over N_{ω_1} . The resulting model, $N_{\omega_1}[Coll(\omega, \alpha)]$, still contains all ordinals countable in \mathbb{V} , and also knows that α is countable, so we can finally reflect the statement $\delta(x_G) \leq \alpha$ to get that $N_{\omega_1}[coll(\omega, \alpha)] \models \delta(x_G) \leq \alpha$ and

$$N_{\omega_1}[coll(\omega, \alpha)] \models A_{x_G} \text{ Borel}.$$

Note that in N_{ω_1} , α is under a measurable cardinal, hence by Martin-Solovay's theorem, collapsing α over N_{ω_1} preserves Σ_3^1 statements. Proposition 3.3 then assures that $N_{\omega_1} \models A_{x_G} \text{ Borel}$. Since N_{ω_1} is elementarily equivalent to N , we have so far shown that

$$N \models A_{x_G} \text{ Borel},$$

which means that $N \models \delta(x_G) \leq \alpha$ for some $\alpha < \omega_1^N = \omega_1^M$. Another use of the elementary equivalence of N and N_{ω_1} proves that $N_{\omega_1} \models \delta(x_G) \leq \alpha$, from which Σ_2^1 absoluteness guarantees

$$\mathbb{V} \models \delta(x_G) \leq \alpha < \omega_1^M.$$

Taking B to be the set of M -generics concludes the proof. Notice that if I is ccc, B is co- I . \square

3.2. Rectangular Borel canonization of provably ccc ideals. We follow Stern's definitions and results from [18]. By an α -Borel code, for α a not necessarily countable ordinal, we mean a well founded tree on α whose maximal points are associated with basic open sets. An α -Borel code naturally codes a set generated from basic open sets by unions and intersections of length at most α . If α is countable, the set coded by an α -Borel code is Borel.

For a countable ordinal $\gamma < \omega_1$, $L[\gamma]$ stands for $L[a]$ where a codes a well order of ω of order type γ .

Theorem 3.10. (Stern [18]) *If A is $\Pi_\gamma^0 \cap \Pi_1^1(z)$, then $L[z, \gamma]$ has an $\omega_\gamma^{L[z, \gamma]}$ -Borel code for A .*

Proposition 3.11. *Let A be a $\Sigma_1^1(z)$ subset of the plane with Π_γ^0 sections. Let I be a σ -ideal proper in $L[z, \gamma]$, and x generic over $L[z, \gamma]$. Then*

$$\delta(x) < \omega_{\gamma+1}^{L[z, \gamma]}.$$

Proof. Since $\mathbb{V} \models A_x$ is Π_γ^0 , using Stern's theorem we know that $L[z, x, \gamma] \models A_x$ is $\omega_\gamma^{L[z, x, \gamma]}$ -Borel. Collapsing $\omega_\gamma^{L[z, x, \gamma]}$ over $L[z, x, \gamma]$, we have:

$$L[z, x, \gamma][Coll(\omega, \omega_\gamma^{L[z, x, \gamma]})] \models A_x \text{ Borel}.$$

ω_1 of the new model is $\omega_{\gamma+1}^{L[z, x, \gamma]}$. Since x is assumed to be $L[z, \gamma]$ -generic and \mathbb{P}_I doesn't collapse cardinals in $L[z, \gamma]$:

$$\omega_{\gamma+1}^{L[z, \gamma]} = \omega_{\gamma+1}^{L[z, \gamma][x]}.$$

Hence there must be an $\alpha < \omega_{\gamma+1}^{L[z, \gamma]}$ such that

$$L[z, x, \gamma][Coll(\omega, \omega_\gamma^{L[z, x, \gamma]})] \models \delta(x) \leq \alpha.$$

Shoenfield's absoluteness concludes the proof. \square

Theorem 3.12. *Assume ω_1 is inaccessible to the reals, and I is ccc in $L[z]$ for any real z . Then I has strong rectangular Borel canonization of analytic sets all of whose sections are Π_γ^0 for some $\gamma < \omega_1$.*

Note that part of the assumption here is that I is defined and a σ -ideal in $L[z]$ for any real z .

Proof. Let A be a $\Sigma_1^1(z)$ set with Π_γ^0 sections. Since I is ccc in $L[z, \gamma]$, the set of generics over $L[z, \gamma]$ is co- I . ω_1 is inaccessible in $L[z, \gamma]$, so that in particular $\omega_{\gamma+1}^{L[z, \gamma]} < \omega_1$. The previous proposition then concludes the proof. \square

4. EXAMPLES AND COUNTEREXAMPLES

The following section elaborates on Borel canonization of equivalence relations in its most general form:

Problem 4.1. Given an analytic equivalence relation E on a Polish space X and a σ -ideal I , does there exist an I -positive Borel set B such that E restricted to B is Borel?

As mentioned before, in general the answer is negative. We will list a few examples and counterexamples we find interesting.

4.1. Non Borel classes and improper ideals.

Example 4.2. (Kanovei, Sabok, Zapletal [14]) Fix K a strictly analytic Borel ideal on ω and define an equivalence relation on $(2^\omega)^\omega$ by:

$$\bar{x}E\bar{y} \Leftrightarrow \{n : \bar{x}(n) \neq \bar{y}(n)\} \in K.$$

Not only that E is non Borel – one can easily show that none of its classes is Borel. Now consider the following σ -ideal I on $(2^\omega)^\omega$: $A \notin I$ if A does not contain a set of the form $\Pi_{n \in \omega} P_n$ for P_n perfect sets. In [14] it is shown that I is a σ -ideal, and even a proper one. However, for any B Borel and I -positive, E can be reduced to $E \upharpoonright_B$ (since B contains a copy of the whole space). In particular, $E \upharpoonright_B$ is not Borel.

The above example clarifies why we assume all classes are Borel, and why that assumption is not redundant even when one assumes the properness of the ideal I .

Example 4.3. Let E be an analytic and non Borel **orbit equivalence relation** on ω^ω . Consider the following σ -ideal : $C \in I$ if there is $B \supseteq C$ Borel such that $E \upharpoonright_B$ is Borel.

- (1) $\omega^\omega \notin I$, since E is non Borel. Trivially enough, $\emptyset \in I$.
- (2) I is downward closed.
- (3) To show that I is σ -closed, consider $\langle C_n : n \in \omega \rangle$ a sequence of sets in I , and let $\langle B_n : n \in \omega \rangle$ Borel such that $B_n \supseteq C_n$ and $E \upharpoonright_{B_n}$ is Borel. Since $\bigcup B_n \supseteq \bigcup C_n$, we will be satisfied showing that $E \upharpoonright_{\bigcup B_n}$ is Borel. That follows easily from Hjorth analysis (see [9, 4]): Let δ_n be some countable ordinal bounding the Hjorth rank on B_n . Then $\sup_n(\delta_n)$ bounds the rank on $\bigcup_n B_n$, hence $E \upharpoonright_{\bigcup B_n}$ is Borel.
- (4) There is no Borel canonization of E with respect to I : if B is Borel and I positive then by the very definition of I , $E \upharpoonright_B$ is non Borel.

Remark 4.4. In 3 we have used the fact that E is an orbit equivalence relation. We conjecture it is not necessarily true for general analytic and non Borel equivalence relations.

In example 4.3, Borel canonization fails although all classes are Borel. However, the σ -ideal considered here is non proper. The sets $\mathcal{A}_\alpha = \{x : \delta(x) \leq \alpha\}$, where δ is the Hjorth rank, are Borel and I -small. Hence a generic element will have rank greater or equal than ω_1 , clearly collapsing ω_1 . That example thus indicates the necessity of assuming the properness of the σ -ideal I .

4.2. Perfect set properties of equivalence relations.

Definition 4.5. Let E be an equivalence relation on a Polish space X . E has perfectly many classes if there is a perfect set $P \subseteq X$ of pairwise inequivalent elements.

One of the most well known results in the study of equivalence relations in set theory is the following theorem due to Silver:

Theorem 4.6. (Silver) *Let E be a coanalytic equivalence relation on a Polish space X . Then either E has countably many classes, or it has perfectly many classes.*

Silver's theorem fails for analytic equivalence relations:

- (1) For $x, y \in LO$ linear orders, let

$$xE_{\omega_1}y \Leftrightarrow x, y \notin WO \vee ot(x) = ot(y).$$

Then E_{ω_1} has uncountably many classes – $\langle WO_\alpha : \alpha < \omega_1 \rangle$ and the class of ill orders. However, E_{ω_1} does not have perfectly many classes, as by the boundedness theorem perfect sets in WO will have bounded order type. Note that all but one of the equivalence classes are Borel.

- (2) For $x, y \in \omega^\omega$, let

$$xE_{ck}y \Leftrightarrow \omega_1^{ck(x)} = \omega_1^{ck(y)}.$$

Then E_{ck} is analytic with uncountably many classes. The effective version of the boundedness theorem demonstrates that E_{ck} does not have perfectly many classes. Notice that all the E_{ck} classes are Borel.

- (3) Given a Polish group action (G, X) inducing a non Borel orbit equivalence relation, let

$$xE_\delta y \Leftrightarrow \delta(x) = \delta(y),$$

where δ is the Hjorth rank (as in [9, 4]). E_δ is analytic with uncountably many classes. It does not have perfectly many classes – otherwise we could have ccc forced $\neg CH$, and use Shoenfield's absoluteness to get in the generic extension a perfect set of size less than the continuum. Notice that here as well, all the E_δ classes are Borel.

Proposition 4.7. *Let E be analytic with uncountably many classes but not perfectly many. Let I_E be the σ -ideal generated by the equivalence classes. Then for any B Borel I_E -positive, $E \restriction_B$ is non Borel.*

Proof. Let B be Borel I_E -positive. Since the equivalence classes are I_E -small, B must intersect uncountably many classes. If $E \restriction_B$ was Borel, Silver's theorem would produce a perfect set of pairwise inequivalent elements – contradicting the assumptions on E . \square

Hence E_{ω_1} , E_{ck} and E_δ together with their induced σ -ideals all serve as counterexamples – the first to Borel canonization of analytic equivalence relations, and the 2nd and 3rd to Borel canonization of analytic equivalence relations with Borel classes. Fortunately, $I_{E_{\omega_1}}$, $I_{E_{ck}}$ and I_{E_δ} are all improper – in fact $\mathbb{P}_{I_{E_{\omega_1}}}$, $\mathbb{P}_{I_{E_{ck}}}$ and $\mathbb{P}_{I_{E_\delta}}$ all collapse ω_1 :

- (1) Given $\gamma < \omega_1$, the set $W_\gamma = \{x : \forall k \text{ } ot(x \restriction_k) < \gamma\}$ is in $I_{E_{\omega_1}}$. The generic real must avoid all of them, hence its well founded part has order type greater or equal than ω_1 – thus collapsing ω_1 .
- (2) Let x_G be the generic real added by forcing with $\mathbb{P}_{I_{E_{ck}}}$. Then $\omega_1^{ck(x_G)} \geq \omega_1$.
- (3) Let x_G be the generic real added by forcing with $\mathbb{P}_{I_{E_\delta}}$. Then $\delta(x_G) \geq \omega_1$.

This is no coincidence: In [5] we show that for E analytic with uncountably many classes but not perfectly many, I_E is improper.

When considering the above equivalence relations with proper ideals, Borel canonization is trivially found – we show it for E_{ck} , proofs for the other two are almost the same.

Example 4.8. Consider E_{ck} and a proper ideal I . Since \mathbb{P}_I doesn't collapse ω_1 , there must be some $\alpha < \omega_1$ such that $\{x : \omega_1^{ck(x)} = \alpha\}$ is I -positive. E_{ck} restricted to that Borel set is trivial.

4.3. Borel canonization of Δ_2^1 sets. We end this section considering equivalence relations which are less definable. By Borel canonization we still mean – “ $E \restriction_B$ is Borel for some B Borel I -positive set” or “ $A \cap (B \times B)$ is Borel for some B Borel I -positive set”, etc.

Theorem 4.9. [3] *In L , there is a countable Δ_2^1 equivalence relation that does not have perfectly many classes. In particular, in L σ -ideals do not have Borel canonization of Δ_2^1 equivalence relations with Borel classes.*

Proof. In L , consider the following equivalence relation:

$$xEy \Leftrightarrow (\forall \alpha \text{ admissible } x \in L_\alpha \Leftrightarrow y \in L_\alpha).$$

Since the constructibility rank of x and the admissibility of ordinals are decided by a countable model and by all countable models, E is a Δ_2^1 equivalence relation. All E classes are countable, since all L'_α s are. We will show that any perfect tree T must have two equivalent elements.

Let $T \in L$ be perfect, and let α be such that $T \in L_\alpha$. Let β be the first admissible ordinal greater than α such that L_β has a real not in L_α . Using [3] fact 9.5, L_α is countable in L_β . Since T has uncountably many branches in L_β , there must be

$$x \neq y \in [T] \cap L_\beta$$

that are not in L_α . It follows that x and y are equivalent. □

The following proposition is weaker, but its proof is easier:

Proposition 4.10. *In L , σ -ideals do not have square Borel canonization of Δ_2^1 sets with Borel sections.*

Proof. Denote by $<_L$ the Δ_2^1 well order of the reals in L , whose horizontal and vertical sections are all Borel. The set $<_L$ does not have square Borel canonization with respect to any σ -ideal. This is because $<_L$ restricted to an uncountable set is an order of length ω_1 , hence by the boundedness theorem for analytic well founded relations, it cannot be analytic. □

All the above can be done in $L[z]$ for z real, and in any model in which $\mathbb{R}^{L[z]} = \mathbb{R}$ for some $z \in \mathbb{R}$.

5. COUNTEREXAMPLES TO RECTANGULAR BOREL CANONIZATION

Counterexamples are implicit in [11]:

Example 5.1. [11] Consider the meager ideal and theorem 1.7. For that ideal, rectangular Borel canonization and strong rectangular Borel canonization are equivalent (see remark 1.5). Hence theorem 1.7 provides counterexamples when not all Σ_2^1 sets have the Baire property. The same is true for the null ideal.

Proposition 5.2. *In L , proper ideals do not have rectangular Borel canonization of analytic sets with Borel sections. The same is true for $L[z]$ where z is a real.*

Proof. The argument is based on example 2.3.5 of [19]. Working in L , let

$$(x, y) \in A \Leftrightarrow x \in L_{\omega_1^{ck(y)}}.$$

The set A is coanalytic with Borel vertical sections, since given $x \in L_\alpha$ and α minimal with that property,

$$A_x = \{y : x \in L_{\omega_1^{ck(y)}}\} = \{y : \omega_1^{ck(y)} \geq \alpha\},$$

which is Borel. By way of contradiction, fix B Borel I -positive such that $A \cap (B \times \omega^\omega)$ is Borel. Using \mathbb{P}_I -uniformization (2.3.4 of [19]) there exists $C \subseteq B$ Borel I -positive and $f : C \rightarrow \omega^\omega$ Borel such that $f \in L$ and $f \subseteq A$. Let $x_G \in C$ be a Sacks real over L . By analytic absoluteness, $L[x_G] \models f \subseteq A$, and in particular, $(x_G, f(x_G)) \in A$, contradicting the fact that x is not constructible. \square

Let A be an analytic subset of the plane and $B \subseteq \omega^\omega$ Borel I -positive subset of reals such that $A \cap (B \times \omega^\omega)$ is Borel. Then using Shoenfield's absoluteness, $\mathbb{P}_I \Vdash A \cap (B \times \omega^\omega)$ Borel, and in particular

$$B \Vdash A_{x_G} \text{ Borel}.$$

Hence an ideal I such that \mathbb{P}_I adds a non Borel section is a counterexample to rectangular Borel canonization. We now show that even under mild large cardinal assumptions, there might exist such an ideal which is ccc:

Fact 5.3. *If ω_1 is inaccessible to the reals and is not Mahlo in L , then there is a ccc forcing adding a real x such that $\omega_1^{L[x]} = \omega_1$.*

For the proof, see theorem 6 of [1].

Proposition 5.4. *If \mathbb{P} is a ccc forcing adding a real x , then there is a ccc ideal I such that $\mathbb{V}[x] \subseteq \mathbb{V}^\mathbb{P}$ is a \mathbb{P}_I extension and x is the \mathbb{P}_I generic real.*

Proof. Fix τ a \mathbb{P} -name for the real x . For B Borel, define

$$B \in I \Leftrightarrow \mathbb{P} \Vdash \tau \notin B.$$

I is a σ -ideal (in fact, a σ -ideal on Borel sets which generates a σ -ideal). We claim that it is ccc. Let $\langle B_\alpha : \alpha < \omega_1 \rangle$ be an antichain of I -positive sets, which is, for $\alpha_1 \neq \alpha_2$,

$$\mathbb{P} \Vdash \tau \notin (B_{\alpha_1} \cap B_{\alpha_2}).$$

Fix $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash \tau \in B_\alpha$. Then $\langle p_\alpha : \alpha < \omega_1 \rangle$ must be an antichain, hence countable, as we have hoped.

In $\mathbb{V}^\mathbb{P}$, the generic x , as a realization of τ , avoids all Borel I -small sets of the ground model, hence it is \mathbb{P}_I generic over \mathbb{V} . Thus $\mathbb{V}[x]$ is the promised \mathbb{P}_I extension. \square

Theorem 5.5. *If ω_1 is inaccessible to the reals and is not Mahlo in L , then there is a ccc ideal I not having rectangular Borel canonization of analytic sets with Borel sections. Moreover, $\mathbb{P}_I \Vdash A_{x_G}$ non Borel for some A analytic with Borel sections.*

Proof. For every x , in $L[x]$ there exists a $\Pi_1^1(x)$ uncountable set with no perfect subset. Fix $\Phi(x)$ a $\Pi_1^1(x)$ formula defining that set. Then in any universe \mathbb{V} , $\Phi(x)$ defines a subset of $L[x]$ of size $\omega_1^{L[x]}$ with no perfect subset. Moreover, the definition is uniform – there is a Π_1^1 formula $\Psi(x, y)$ such that for every x ,

$$\{y : \Psi(x, y)\}$$

is a subset of $L[x]$ of size $\omega_1^{L[x]}$ with no perfect subset.

Consider the subset of the plane defined by Ψ . The vertical sections of Ψ are either countable or strictly coanalytic – since we assume ω_1 is inaccessible to the reals, they are all countable and in particular Borel. Use the forcing of proposition 5.3 to obtain a ccc extension $\mathbb{V}^{\mathbb{P}}$ with $x \in \mathbb{V}^{\mathbb{P}}$ such that $\omega_1^{L[x]} = \omega_1$. Use the previous proposition to construct a ccc ideal I such that

$$\mathbb{V}[x] = \mathbb{V}^{\mathbb{P}^I}.$$

Obviously, $\mathbb{P}_I \Vdash \omega_1^{L[x_G]} = \omega_1$ for x_G its generic real. In particular, Ψ has a new section which is non Borel, and rectangular Borel canonization fails. \square

Remark 5.6. The reader is encouraged to compare the above example with the positive results of previous sections. When doing so, note that I is not even defined in L – its definition requires a club in ω_1 of ordinals which are singular cardinals in L .

Corollary 5.7. *Rectangular Borel canonization for ccc ideals implies that ω_1 is inaccessible to the reals and Mahlo in L .*

Proof. By theorem 2.11, Hechler ideal has rectangular Borel canonization if and only if Σ_2^1 sets are Hechler measurable. In [2] it is shown that measurability of Σ_2^1 sets with respect to the Hechler ideal is equivalent to ω_1 being inaccessible to the reals. To see that ω_1 is Mahlo in L , use the previous theorem. \square

The case of square Borel canonization is different – for A analytic and B Borel, if $A \cap (B \times B)$ is Borel, then $\mathbb{P}_I \Vdash A \cap (B \times B)$ is Borel, hence

$$B \Vdash (A_{x_G} \cap B) \text{ is Borel.}$$

In order to construct a counterexample, we can try and find A and I such that no B Borel I -positive forces the Borelness of $A_x \cap B$:

Problem 5.8. Let Ψ and I be as in theorem 5.5, and let A be the coanalytic subset of the plane defined by Ψ . Can we find $B \in \mathbb{P}_I$ such that $B \Vdash (A_{x_G} \cap B) \text{ is Borel}$?

5.1. Non absoluteness of “All classes are Borel”. The previous example shows that for A an analytic subset of the plane, the property “all vertical sections of A are Borel” can be forced false by a ccc ideal. The same applies for analytic equivalence relations:

Proposition 5.9. *There is an analytic equivalence relation E such that:*

- (1) *If ω_1 is inaccessible to the reals and is not Mahlo in L , then all E classes are Borel and there is a ccc ideal I such that*

$$\mathbb{P}_I \Vdash [x_G] \text{ is non Borel.}$$

- (2) *If ω_1 is inaccessible to the reals, then all E classes are Borel, while in L there is a non Borel class.*

Proof. We use a variation of the example introduced by theorem 5.5.

Let $\Psi(x, y)$ be as in theorem 5.5 – a Π_1^1 formula whose vertical sections are subsets of $L[x]$ of size $\omega_1^{L[x]}$ with no perfect subset. Let

$$(x_1, y_1)E(x_2, y_2) \Leftrightarrow (x_1 = x_2) \wedge (((\neg\Psi(x_1, y_1) \wedge \neg\Psi(x_2, y_2)) \vee (y_1 = y_2))).$$

E is an analytic equivalence relation, and the equivalence class of (x_0, y_0) is either a singleton or

$$\{(x_0, y) : \neg\Psi(x_0, y)\}.$$

Hence if $\neg\Psi(x_0, y_0)$, $[(x_0, y_0)]_E$ is Borel if and only if $\omega_1^{L[x_0]} < \omega_1$.

The 1st clause then follows using the forcing notion introduced in the previous subsection, while the 2nd clause is obvious. \square

Remark 5.10. Failure of downward absoluteness of “all classes are Borel” follows from ZFC alone: In L , fix A a coanalytic uncountable set without a perfect subset, and let

$$xEy \Leftrightarrow (x = y) \vee (x, y \notin A).$$

The analytic equivalence relation E has a non Borel class, but after collapsing ω_1 over L , all its classes become Borel.

Problem 5.11. The nature of the above examples raises the following questions:

- (1) Is there an analytic equivalence relation with Borel classes in L but non Borel classes under large cardinal assumptions?
- (2) Can we prove the failure of upward absoluteness of “all classes are Borel” without using the consistency of an inaccessible cardinal?

We end this section by computing the complexity of various properties discussed in this paper, the most important of them are “section A_x is Borel” and “the rank of x is less than $ot(f)$ ”:

Proposition 5.12. *In what follows, we say that a property is $\Pi_\alpha^1 / \Sigma_\alpha^1$ if it is provably $\Pi_\alpha^1 / \Sigma_\alpha^1$, which is, there is a lightface $\Pi_\alpha^1 / \Sigma_\alpha^1$ formula equivalent to the property in every model of ZFC . Then, assuming $con(ZFC + Inaccessible)$ and given A a subset of the plane:*

- (1) *If A is Σ_1^1 , “all sections of A are countable” is Π_1^1 .*
- (2) *If A is Π_1^1 , “all sections of A are countable” is Π_4^1 , and is neither Π_3^1 nor Σ_3^1 .*
- (3) *If A is Π_1^1 , “all sections of A do not contain a perfect set” is Π_2^1 , hence is absolute between generic extensions.*
- (4) *If A is Σ_1^1 or Π_1^1 , “all sections are Borel” is Π_4^1 , and is neither Π_3^1 nor Σ_3^1 . The same is true when A is an equivalence relation.*
- (5) *If A is Σ_1^1 or Π_1^1 and $\alpha < \omega_1^{ck}$, “all sections are Π_α^0 ” is Π_4^1 , and is neither Π_3^1 nor Σ_3^1 .*
- (6) *If A is Σ_1^1 or Π_1^1 , the set $\{x : A_x \text{ is Borel}\}$ is Σ_3^1 , and not Π_3^1 .*
- (7) *If A is Σ_1^1 and δ is some rank associated with it, then $\{(x, f) : f \in WO, \delta(x) \leq f\}$ is not Σ_2^1 .*

Most of the above can be relativized.

Proof. For (1), recall that a $\Sigma_1^1(x)$ set is countable if and only if all its elements are hyperarithmetic in x . For (2), use theorem 5.5, proposition 5.9 and Shoenfield’s absoluteness. Regarding (6), note that if “ A_x is Borel” had been Π_3^1 then “all sections are Borel” would have been Π_3^1 as well. In (7), a Σ_2^1 definition for this set will produce a Σ_2^1 definition of “ A_x is Borel”. \square

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